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LETTER TO THE EDITOR

Remanence effects for spin glasses with sequential dynamics: exact results

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Abstract. The remanent magnetization and the remanent energy are calculated exactly for the $\pm J$ spin glass in one dimension with random field and on Cayley trees for sequential dynamics at zero temperature. The method used is an iteration scheme which classifies different kinds of spins. In one dimension a value of $\frac{11}{60}$ is found for the remanent magnetization. As expected, the result for the Cayley tree strongly depends on the boundary conditions. The tree with branching number 2 at its 'equilibrium' (i.e. the distribution of the boundary is the same as for the bulk) has a remanent magnetization of $\frac{1}{7}(-33+25\sqrt{2}) \approx 0.336$.

The dynamical behaviour of spin glasses is governed by a huge (exponential) number of metastable states [1]. One manifestation of this fact is the occurrence of remanence effects because the system becomes trapped in one of the metastable states before attaining the thermodynamic equilibrium (e.g. the ground state at zero temperature).

The analytical investigation of the most interesting quantity in this context, the remanent magnetization, is rather difficult because the dynamical equations have to be solved completely. For the standard $\pm J$ spin chain at zero temperature ($T=0$) it is known that the remanent magnetization vanishes with a power law $m_{\text{rem}} \sim t^{-1/2}$ [2, 3], whereas random infinite/zero random fields modify this to a stretched exponential decay [4, 5].

Non-trivial exact results with a finite remanent magnetization are known only for the spin-glass chain with continuous distribution of the couplings and vanishing external field at $T=0$ [6–8]. Unfortunately in this system frustration plays a minor role since it is restricted to a certain class of bonds ('weak bonds') and all other couplings can be 'satisfied' simultaneously.

In this letter the results of a new method are reported which yields analytic results for the remanent magnetization and energy under sequential dynamics for $\pm J$ spin glasses in one dimension with (random) external fields and for the Cayley tree with arbitrary (even) branching number at $T=0$. (The update direction in the Cayley tree is from the leaves to the root.) Details of the method and the complete derivation of the results will be published in a forthcoming paper [9].

The basic assumption of the method is that the local field of all spins will never vanish (i.e. the fields are different from zero with probability one). If there is a finite probability for the local fields to vanish, the question arises of how these spins should be updated. A random choice of the new state independently at every time step

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introduces additional noise which gives rise to a slowing down of the remanent magnetization. This noise is not correlated in time and therefore corresponds to an annealed situation. A quenched situation would be present if for every spin at the beginning a favoured but random direction would be chosen. But this is equivalent to the introduction of an (arbitrarily) small quenched random field and to this case the method applies. In the case of the Cayley tree with branching number two, the magnetic field is replaced by the third next neighbour.

The essential key to the solution is to classify the spins into different (disjoint) sets according to their *possibility* to flip. The method is explained easiest for the Cayley tree with branching number 2 and random binary couplings $J_{ij} = \pm J$ and fields $h_i = \pm h$ ($h = J$) with independent probabilities $\frac{1}{2}$. Since the update direction is from the leaves to the root (see figure 1), every spin (e.g. σ_0) has two predecessors (σ_1, σ_2) which will be updated before the spin itself. These two spins will cause a local field with absolute value of either 0 or $2J$. If it is 0, the spin will align to the local field produced by the third (next) neighbour (σ_3); in the other case ($2J$) knowledge of this field is not necessary, because it is lower than $2J$.

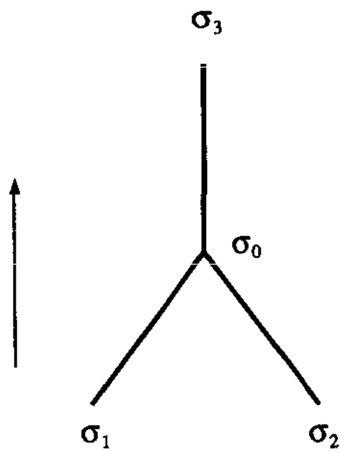

Figure 1. The local structure of a tree with branching number two showing the predecessors σ_1, σ_2 and the successor σ_3 of spin σ_0. The update direction is from the leaves to the root (upwards along the arrow).

One then can write down iteration equations for the probabilities of the different kinds of spins occurring, with the expectation that one will get a closed system. Every spin type gives one equation and therefore the applicability of the method is called into question if the number of different spin types becomes too large. Fortunately in the case of the systems mentioned above only three different kinds of spin occur: fixed (f), single-flip (sf) and multi-flip (mf) spins with probabilities P_f, P_{sf} and P_{mf}, respectively. The fixed spins are completely determined by the local field of the predecessors ($\pm 2J$). A multi-flip spin has a vanishing field from the predecessors for all times and thus will align according to the field of the successor. A single-flip has also a vanishing field from the predecessor but only until it flips for the first time. After this flip it is fixed through the field from the predecessors which, after their update, has an absolute value of $2J$.

The iteration equations at level $l+1$ of the tree, averaged over all configurations of the couplings ($P(J=\pm J)=\frac{1}{2}$), are then given by

$$P_{\mathrm{f}}(l+1)=1-P_{\mathrm{f}}(l)+\tfrac{1}{2}P_{\mathrm{f}}^2(l) \tag{1}$$

$$P_{\mathrm{sf}}(l+1)=P_{\mathrm{f}}(l)(P_{\mathrm{mf}}(l)+\tfrac{1}{2}P_{\mathrm{sf}}(l)) \tag{2}$$

$$P_{\mathrm{mf}}(l+1)=\tfrac{1}{2}P_{\mathrm{f}}(l)(1-P_{\mathrm{mf}}(l)) \tag{3}$$

with the normalization

$$P_{\mathrm{f}}(l)+P_{\mathrm{sf}}(l)+P_{\mathrm{mf}}(l)=1 \tag{4}$$

for all l. This means that every spin belongs to exactly one of the three sets. For finite l these equations must be analysed numerically. The long-time limit, which does not depend on the initial conditions at $l=0$ ($\equiv$ boundary of the tree), can be calculated directly yielding the 'equilibrium' (bulk) probabilities

$$P_{\mathrm{f}}(\infty)=2-\sqrt{2}\simeq 0.5858 \tag{5}$$

$$P_{\mathrm{sf}}(\infty)=\tfrac{2}{7}(-5+4\sqrt{2})\simeq 0.1877 \tag{6}$$

$$P_{\mathrm{mf}}(\infty)=\tfrac{1}{7}(3-\sqrt{2})\simeq 0.2265. \tag{7}$$

For the probability distribution of the total remanent magnetization $M(l)$ of the tree with l levels (branching number 2) one can write down a generating function from which one can deduce the expectation values in the standard way.

The remanent magnetization is defined in the standard way as the overlap of the initial and the final state of the spins

$$m_{\mathrm{rem}}=\lim_{t\to\infty}\langle\sigma(0)\sigma(t)\rangle \tag{8}$$

where the brackets mean the average over the coupling configurations. Due to the gauge invariance of the (even) distribution of the couplings the initial condition of the spins can be arbitrary. For simplicity all spins are set to $+1$.

The expectation value of the total remanent magnetization fulfils in this case the iteration equation

$$M(l+1)=2M(l)+1-P_{\mathrm{f}}(l)(1+\tfrac{1}{2}\mathcal{N}(l)) \tag{9}$$

where an auxiliary quantity, the mean number of successive single-flip spins, enters. The iteration for this quantity is given by

$$\mathcal{N}(l+1)=P_{\mathrm{f}}(l)[1-P_{\mathrm{f}}(l)+\tfrac{1}{2}(\mathcal{N}(l)-P_{\mathrm{sf}}(l))]. \tag{10}$$

The long-time limit of $\mathcal{N}$ is

$$\mathcal{N}(\infty)=\sqrt{2}\,P_{\mathrm{sf}}(\infty)=\tfrac{2}{7}(8-5\sqrt{2})\approx 0.2654. \tag{11}$$

The solution of (9) is formally given by

$$M(l)=\sum_{i=0}^{l-1}2^{l-i-1}[1-P_{\mathrm{f}}(i)(1+\tfrac{1}{2}\mathcal{N}(i))]. \tag{12}$$

From this solution it can be seen that the most important terms are those near the boundary. This is due to the fact that the number of the boundary spins is of the same order as the total number of spins. In order to calculate the remanent magnetization per spin in the infinite tree, defined by

$$m_{\mathrm{rem}}=\lim_{l\to\infty}\frac{M(l)}{2^l-1} \tag{13}$$

one has to calculate (12) numerically. If, for example, the boundary spins are all fixed, one has $m_f \approx 0.2056$, while for free boundary spins $m_{mf} \approx 0.6028$. A simple argument shows that these two values are correlated via the equation $2m_{mf} - 1 = m_f$. Only if the boundary has the same distribution of spins as the bulk, can the remanent magnetization be calculated directly to give

$$m_{eq} = \tfrac{1}{7}(-33 + 25\sqrt{2}) \approx 0.3365. \tag{14}$$

This value is comparable to the result $\frac{1}{3}$ of the 1D model with continuous couplings and zero field. With the same method the remanent energy can be calculated via the iteration equation

$$E(l+1) = 2E(l) + P_f(l)(2 - P_f(l)) - 2(1 + \mathcal{N}(l)) \tag{15}$$

which yields for fixed and free boundary spins $e_f \approx -1.227$ and $e_{mf} \approx -1.614$. Again these values are correlated via $2e_{mf} = e_f - 2$. For the equilibrium tree the exact result

$$e_{eq} = -\tfrac{2}{7}(30 - 17\sqrt{2})J \approx -1.702J \tag{16}$$

can be derived. This last value is lower than the other two whereas the corresponding value for the magnetization lies between the other two.

The method is not restricted to the branching number two of the Cayley tree. In the case of random couplings ± 1 every tree with an even branching number N always has non-vanishing local fields and thus satisfies the above-mentioned condition. The iteration equation for the fixed spins, for example, reads

$$P_f^N(l+1) = 1 - \frac{1}{2^N}\binom{N}{N/2}[P_f^N(l)(2 - P_f^N(l))]^{N/2}. \tag{17}$$

The right-hand part of the equation shows that, for $N \to \infty$, the probability of a fixed spin becomes 1. Therefore the remanent magnetization for the equilibrium tree is 0 for infinite N. This result is not surprising since the tree contains no loops, which would give rise to a non-trivial behaviour as in the SK model [10–12].

In one dimension the random external field $P(h_i = \pm h) = \frac{1}{2}$ plays the role of the third next neighbour in the Cayley tree. The randomness of the field is necessary in order to guarantee that the initial state of the spins is not correlated to the field. Alternatively one could take a random initial spin configuration and all external fields fixed (e.g. up). On the other hand the absolute value of the field should be less than $2J$ otherwise the spins will always be parallel to the fields.

The situation here is much simpler because the boundary effects are negligible and the remanent quantities take on unique values. The iteration equations for the auxiliary quantities and their limiting values can be written as

$$P_f^{1D}(l+1) = \tfrac{1}{2} \tag{18}$$

$$P_{sf}^{1D}(l+1) = \tfrac{1}{4}(1 - P_{sf}^{1D}(l)) \xrightarrow{l\to\infty} \tfrac{1}{5} \tag{19}$$

$$P_{mf}^{1D}(l+1) = \tfrac{1}{4}(\tfrac{3}{2} - P_{mf}^{1D}(l)) \xrightarrow{l\to\infty} \tfrac{3}{10} \tag{20}$$

$$N^{1D}(l+1) = \tfrac{1}{4}(1 + \mathcal{N}^{1D}(l) - P_{sf}^{1D}(l)) \xrightarrow{l\to\infty} \tfrac{4}{15}. \tag{21}$$

One should notice that $P_f^{1D}(l)$ does not depend on l, independent of the initial condition. The iteration equations for M^{1D} and E^{1D}

$$M^{1D}(l+1) = M^{1D}(l) + \tfrac{1}{4}(1 - \mathcal{N}^{1D}(l)) \tag{22}$$

$$E^{1D}(l+1) = E^{1D}(l) - 1 - \mathcal{N}^{1D}(l) \tag{23}$$

now have the unique solutions per spin

$$m^{1D}(\infty) = \tfrac{11}{60} \approx 0.1833 \tag{24}$$

$$e^{1D}(\infty) = -\tfrac{19}{15}J \approx -0.1267J. \tag{25}$$

Obviously the value of the remanent magnetization is much lower than all the values of the Cayley tree shown above and also than that of the 1D model with continuous couplings and vanishing field.

Although the considerations above were made for binary couplings $J_{ij} = \pm J$ and fields $h_i = \pm h$ $(h = J)$ there is a much larger range of validity. If, for example, the absolute value of one of the three couplings of a spin in the tree is always larger than the sum of the other two, this spin will, at zero temperature, align to the local field, caused by the 'dominant' coupling. Therefore it is completely decoupled from the other two neighbours. Although this situation can also be treated with the same method, the more interesting case is the one without a dominant coupling. The easiest way to ensure this is that every triple of adjacent couplings of a spin fulfils the following inequality

$$|J_1| < |J_2| + |J_3|. \tag{26}$$

Certainly this inequality should hold also for every permutation of the Js. In the 1D case one coupling has to be replaced by the external field.

The reason why the method only works for sequential update and chains or Cayley trees is due to the fact that it is based on an iteration scheme which depends on the update sequence. Nevertheless it should be possible to generalize it to include at least parallel update. Random sequential update and finite temperatures are much more difficult to treat since any kind of randomness increases the problems of the calculation drastically.

Another remarkable result which was not mentioned in the considerations above is that even the multi-flip spins change their directions at most twice during the descent into a metastable state. This does not mean that after two runs through the system a metastable state is reached. The maximum number of runs is given by the total number of spins. A numerical study of the SK model shows that about 90% of the spins change their direction at most twice until the system is frozen in. Therefore the remanent magnetization is determined mainly by those spins which change their direction only a few times.

The method described above can serve as a good starting point in order to try to study more complicated situations like the honeycomb lattice in two dimensions where every spin also has three next neighbours. Perhaps the ideas can be generalized to include, at least approximately, the SK model itself, where the numerical results cannot give a conclusive answer [10–12].

Another interesting question concerns the introduction of asymmetric couplings, since numerical investigations of the asymmetric SK model show a transition from a vanishing to a finite remanent magnetization as a function of the degree of the asymmetry [13, 14]. But even in 1D systems with asymmetric couplings, analytical methods are very difficult to apply. A detailed investigation of the asymmetric $\pm J$ system with zero field is the subject of a previous paper [15] while the finite field case will be published in a forthcoming one [9].

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